Matrix Operations

(Sections 2.1 and 2.3)

Notation: If A is an $m \times n$ matrix, we can write

$A = (A_{i,k})$ where $1 \leq i \leq m$ and $1 \leq k \leq n$ and

$A_{i,k}$ is a real number that occurs in the i^{th} row and k^{th} column of A.

Example 1:

Let $A = \begin{bmatrix} -1 & 6 & 15 \\ 38 & 2 & 4 \end{bmatrix}$

Find $A_{1,1}$, $A_{2,3}$, $A_{1,2}$.

Solution: $A_{1,1}$ = entry in the first row, first column of A

$= -1$

$A_{2,3} = 4$

$A_{1,2} = 6$

Matrix Addition

Let $A = (A_{i,k})$, $B = (B_{i,k})$ be $m \times n$ matrices. We add A and B by adding their corresponding entries.

So

$$(A+B)_{i,k} = A_{i,k} + B_{i,k}$$

for all $1 \le i \le m$, $1 \le k \le n$.

Note: A and B have to have the same dimensions in order to add them.

Example 2: Let $A = \begin{bmatrix} 1 & 0 & -5 \\ 3 & 8 & 7 \end{bmatrix}$

$B = \begin{bmatrix} 2 & 9 & 4 \\ 13 & 19 & -2 \end{bmatrix}$

Compute $A+B$.

Solution: $A+B = \begin{bmatrix} 1+2 & 0+9 & -5+4 \\ 3+13 & 8+19 & 7+(-2) \end{bmatrix}$

$= \begin{bmatrix} 3 & 9 & -1 \\ 16 & 27 & 5 \end{bmatrix}$ ✓

Scalar Multiplication

If $A = (A_{i,j})$ is an $m \times n$ matrix and c is scalar, we multiply A by c via multiplying every entry of A by c:

$$(cA)_{i,k} = c \cdot A_{i,k}$$

for all $1 \leq i \leq m$, $1 \leq k \leq n$.

Example 3: Let $A = \begin{bmatrix} 8 & -7 \\ 3 & 15 \end{bmatrix}$.

If $c = -19$, compute

$c \cdot A$.

Solution:

$$-19 \cdot A = -19 \cdot \begin{bmatrix} 8 & -7 \\ 3 & 15 \end{bmatrix}$$

$$= \begin{bmatrix} -152 & 133 \\ -57 & -285 \end{bmatrix}$$

Properties

Let A, B, and D be $m \times n$ matrices and let c, f be constants. Then

1) $(A+B)+D = A+(B+D)$

associativity of addition

2) $A+B = B+A$

commutativity of addition

3) $(c \cdot f) \cdot A = c \cdot (f \cdot A)$

associativity of scalar multiplication

4) $c \cdot (A+B) = c \cdot A + c \cdot B$

distributivity of scalar multiplication over addition

5) $(c+f) \cdot A = c \cdot A + f \cdot A$

distributivity of addition (in $\mathbb{R}$) over scalar multiplication

You are free to use any and all of these properties in this class.

Matrix Multiplication

Let A be an $m \times n$ matrix and let B be an $n \times k$ matrix. We matrix multiply A and B using the dot product (matrix-vector multiplication) as follows:

Write $A = \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_m \end{bmatrix}$ where

$\vec{a}_i$ is a row vector in $\mathbb{R}^n$

for all $1 \leq i \leq m$.

Write

$$B = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_k \end{bmatrix}$$

where $\vec{b}_i$ is a column vector in $\mathbb{R}^n$ for all $1 \leq i \leq k$.

$A \cdot B$ is an $m \times k$ matrix where

$$\boxed{(A \cdot B)_{i,t} = \vec{a}_i \cdot \vec{b}_t}$$ (a real number)

Caution: the number of rows in B must equal the number of columns in A for this to work, just like matrix-vector multiplication!

$$\underset{m \times n}{\underbrace{A}} \cdot \underset{s \times t}{\underbrace{B}}$$

equal

Example 4: $(3\times 2)\cdot(2\times 4)$

Let $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 0 & -1 \end{bmatrix}$

and $B = \begin{bmatrix} 4 & -2 & 6 & 10 \\ 1 & 0 & 5 & 7 \end{bmatrix}$

Compute $A \cdot B$ and $B \cdot A$ wherever possible.

Solution: A is 3×2, B is 2×4.

$B \cdot A$ is not possible since $4 \neq 3$.

But $A \cdot B$ is possible.

$AB =$

$$\begin{bmatrix} [1\ -1]\cdot\begin{bmatrix}4\\1\end{bmatrix} & [1\ -1]\cdot\begin{bmatrix}-2\\0\end{bmatrix} & [1\ -1]\cdot\begin{bmatrix}6\\5\end{bmatrix} & [1\ -1]\cdot\begin{bmatrix}10\\7\end{bmatrix} \\ [2\ 3]\cdot\begin{bmatrix}4\\1\end{bmatrix} & [2\ 3]\cdot\begin{bmatrix}-2\\0\end{bmatrix} & [2\ 3]\cdot\begin{bmatrix}6\\5\end{bmatrix} & [2\ 3]\cdot\begin{bmatrix}10\\7\end{bmatrix} \\ [0\ -1]\cdot\begin{bmatrix}4\\1\end{bmatrix} & [0\ -1]\cdot\begin{bmatrix}-2\\0\end{bmatrix} & [0\ -1]\cdot\begin{bmatrix}6\\5\end{bmatrix} & [0\ -1]\cdot\begin{bmatrix}10\\7\end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -2 & 1 & 3 \\ 11 & -4 & 27 & 41 \\ -1 & 0 & -5 & -7 \end{bmatrix} \checkmark$$

In Wolfram Alpha:

Can write

"A . B" or

"A * B"

Note: (noncommutativity) We saw in the previous example that even if $A \cdot B$ makes sense, $B \cdot A$ may not, so $A \cdot B \neq B \cdot A$.

But even when $B \cdot A$ makes sense, it could be the case that $A \cdot B \neq B \cdot A$! So matrix multiplication is, in general, noncommutative.

Example 5: If $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and

$B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, show

$A \cdot B \neq B \cdot A$.

Solution: $A \cdot B =$

$$\begin{bmatrix} [1 \ 0] \cdot \begin{bmatrix} 0 \\ 0 \end{bmatrix} & [1 \ 0] \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ [0 \ 0] \cdot \begin{bmatrix} 0 \\ 0 \end{bmatrix} & [0 \ 0] \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$B \cdot A =$$

$$\begin{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\neq \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = A \cdot B \checkmark$$

Properties

Let A, B, D, E, F, and G be matrices with

A $m \times n$, B and D $n \times k$, and E $k \times m$.

Let c be a scalar

1) $(A \cdot B) \cdot E = A \cdot (B \cdot E)$

associativity of matrix multiplication

2) $(c \cdot A) \cdot B = c \cdot (A \cdot B)$

associativity of scalar multiplication with matrix multiplication

3) $A \cdot (B+D) = A \cdot B + A \cdot D$

$$(B+D) \cdot E = B \cdot E + D \cdot E$$

distributivity of matrix multiplication and addition

These facts are yours to use for the remainder of the course.

Special Matrices

- zero matrix: the $m \times n$ matrix with a zero in every entry is called the zero matrix, and denoted by $O_{m,n}$. If A is an $m \times n$ matrix,

$$A + O_{m,n} = O_{m,n} + A = A$$

If B is $n \times k$ and D is $l \times m$, then

$$O_{m,n} \cdot B = O_{m,k}$$

$$D \cdot O_{m,n} = O_{l,n}$$

If $m = n$, we usually write O_n for $O_{m,n}$.

- Identity matrix: the $n \times n$ matrix I_n with

$$(I_n)_{i,j} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

for all $1 \leq i, j \leq n$.

If A is $n \times k$ and B is $m \times n$,

$$I_n \cdot A = A$$

$$B \cdot I_n = B$$

In particular, if A is $n \times n$,

$$I_n \cdot A = A \cdot I_n = A$$

$$I_1 = 1$$

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{etc.}$$

Observe that

$$(c \cdot I_n) \cdot A = c \cdot (I_n A)$$
$$= c \cdot (A)$$

So this matrix multiplication recovers scalar multiplication for any scalar c.

- Matrix Units: these are the $n \times n$ matrices $(e_{i,j})_{i,j=1}^{n}$

where

$$(e_{i,j})_{k,l} = \begin{cases} 1, & k=i,\ l=j \\ 0, & k \neq i \text{ or } l \neq j \end{cases}$$

In the 2x2 case:

$$e_{1,1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad e_{1,2} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$e_{2,1} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad e_{2,2} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

These are matrices with only one nonzero entry, that is equal to 1.

Rules:

$$e_{1,1} + e_{2,2} + e_{3,3} + \cdots + e_{n,n} = I_n$$

$$e_{i,j} \cdot e_{k,l} = \begin{cases} O_n, & j \neq k \\ e_{i,l}, & j = k \end{cases}$$

for all $1 \leq i,j,k,l \leq n$.

Additive Inverses

If A is an $m \times n$ matrix,

then $-A$ is the $m \times n$ matrix

with, for all $1 \le i, j \le n$,

$$\boxed{(-A)_{i,j} = -A_{i,j}},$$ i.e.,

Negate all the entries of A.

You can see that

$$A + (-A) = 0_{m,n},$$ so

we can subtract matrices!

Q: What about multiplicative inverses?

A: These don't exist in general!

You will not, in general, be able to "divide" matrices.